Quiver representations & Scattering Diagrams

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Application of Mirror Symmetry

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Mirror symmetry

Scattering diagram

Gross
Carl-Pumperla-Siebert

theta function
(broken line)

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theta function
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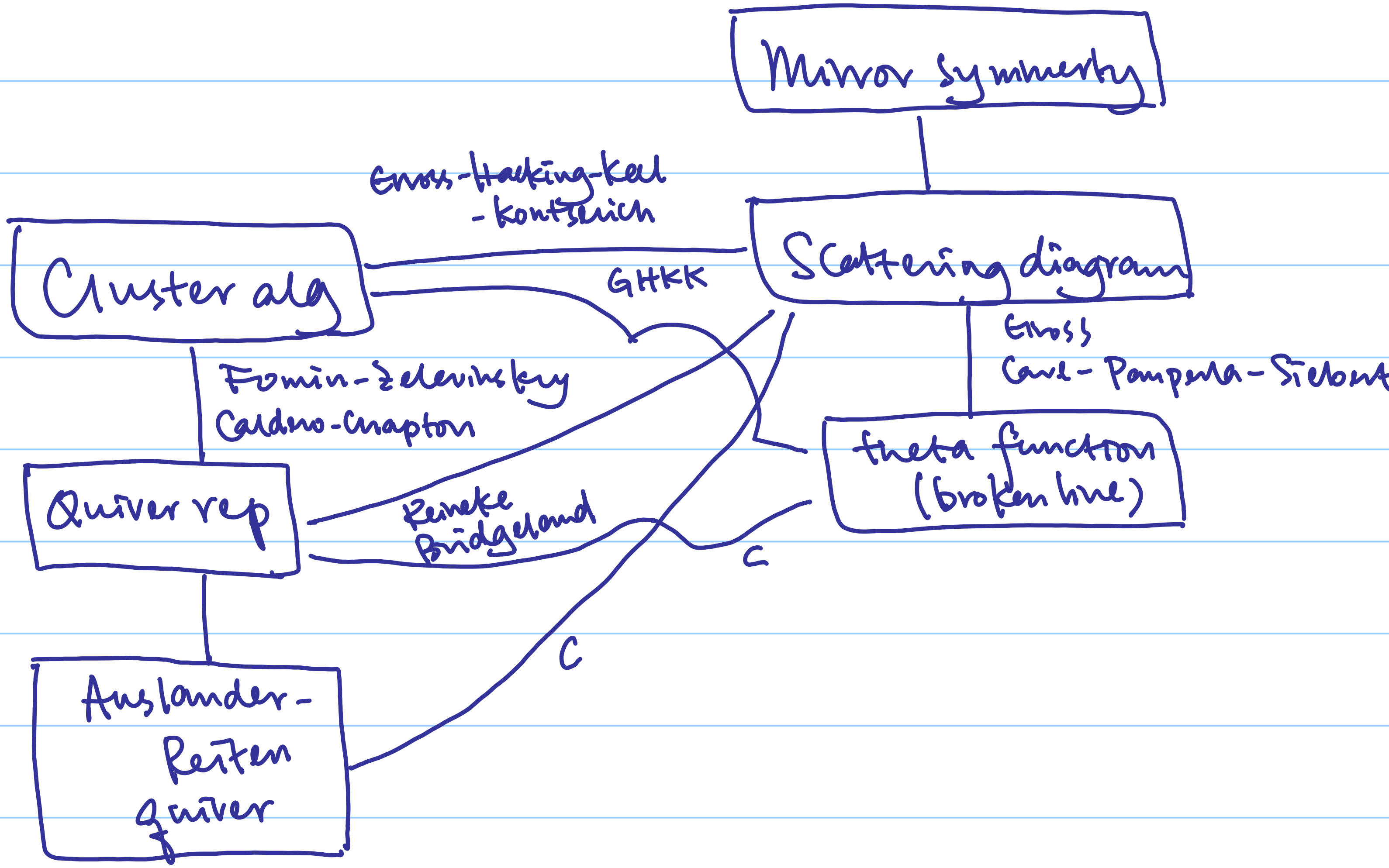
Cluster alg

Fomin-Zelevinsky
Caldero-Chapoton

Quiver rep

Auslander-
Reiten
quiver

Application of Mirror Symmetry



Motivation (Sketch!)

Gross-Siebert program "tropicalize" the SYZ conjecture

Idea: replace special Lagrangian fibration with "good" degeneration.

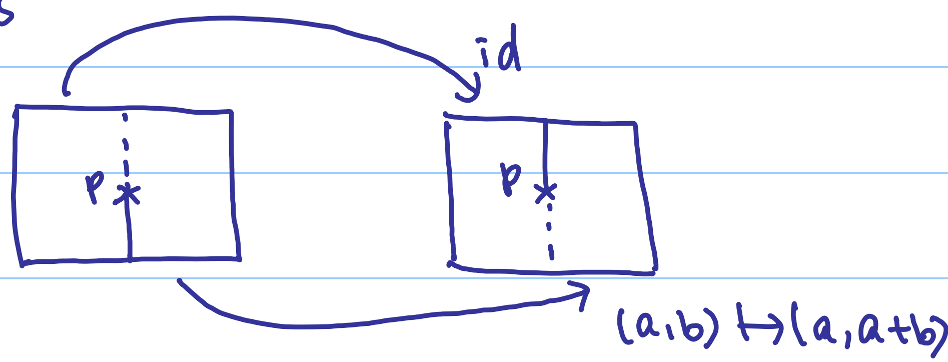
Way: Start with "singular scheme" X_0

provide k-th order deformation order by order.

Do: glue standard thickening "pieces" of X_0

↑ modify standard gluing.

E.g. Two charts



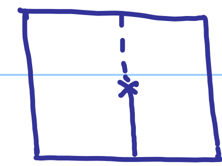
have singularities at p.

Standard gluing is not well-defined.

$z^{(1,0)}(1+z^{(0,-1)}) \mapsfrom z^{(1,0)}$

$z^{(-1,0)}(1+z^{(0,-1)}) \mapsfrom z^{(-1,0)}$

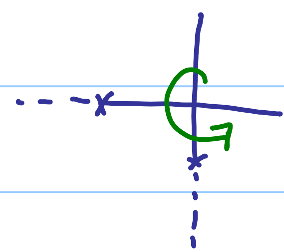
modify by



$z^{(1,0)}(1+z^{(0,-1)}) \mapsfrom z^{(1,0)}$

$z^{(-1,0)}(1+z^{(0,-1)})^{-1} \mapsfrom z^{(-1,0)}$

What happens if

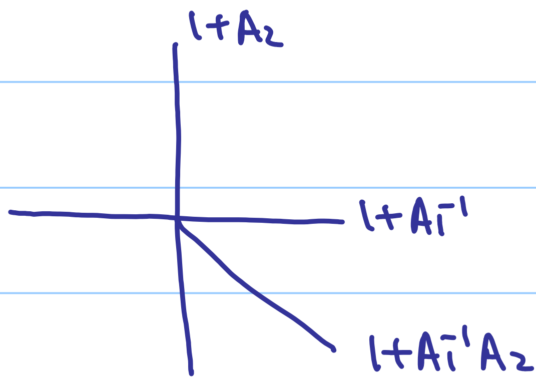


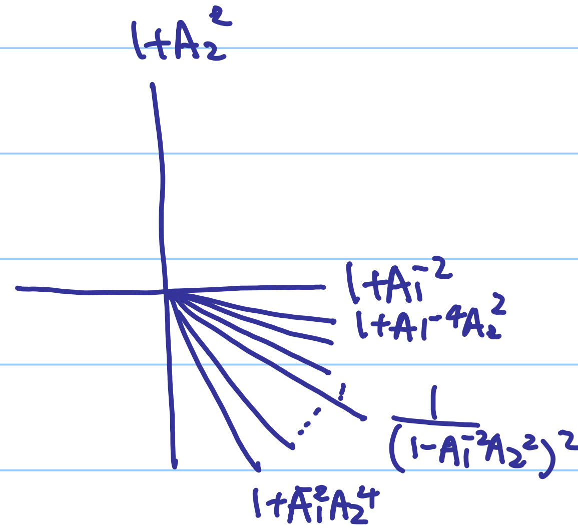
Problem: modify gluing is not enough.

Solution: scattering.

Scattering diagram

E.g.





Note A_i is nothing but x_i

N : lattice of rank n, $M = \mathrm{Hom}(N, \mathbb{Z})$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $M_{\mathbb{R}} = M \otimes \mathbb{R}$

for $m = (m_1, \dots, m_n)$ write $A^m = A_1^{m_1} \cdots A_n^{m_n}$

let $\{\cdot,\cdot\} : N \times N \to \mathbb{Z}$ be a skew-sym bilinear of N

Define
$$p^* : N \to M$$
$$n \mapsto \{n, \cdot\}$$

Assume p^* inj

Wall (d, f_d)

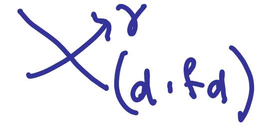
- $d \subseteq M_{\mathbb{R}}$, support of the wall, is a convex rational polyhedral cone of codimension one, $d \subseteq n^{\perp}$ for some $n \in N^+$ (i.e. $n_i \geq 0\ \forall i$)
- $f_d \in \mathbb{C}[A_1, \ldots A_n]$ s.t. $f_d = 1 + \sum_{k \geq 1} c_k A^{k p^*(n)}$ for some $c_k \in \mathbb{C}$

A wall (d, f_d) is called outgoing if $-p^*(n) \in d$

<u>Def</u> A scattering diag. $\mathfrak{D}$ is a collection of walls such that for each $k \geq 0$, the set

$$\{(d, f_d) \mid f_d \not\equiv 1 \bmod (A_1 \ldots A_n)^k\} \text{ is finite.}$$

Wall Crossing



Let γ be a path on $M_{\mathbb{R}}$, and if γ passes a wall (d, f_d)

we define an auto $p_{\gamma,d}: A^m \mapsto A^m f_d^{\langle m, n_0 \rangle}$ where $n_0 = \pm n_d$ s.t.

$\langle n_0, \gamma'(t) \rangle < 0$

If γ passes thru a sequence of walls $d_1, \dots, d_s$,

$$p_{\gamma,\mathfrak{D}} = \prod_{\overrightarrow{\gamma}} p_{\gamma, d_i}$$

<u>Def.</u> $\mathfrak{D}$ is consistent if $p_{\gamma,\mathfrak{D}}$ only depends on the endpt. of γ for any path γ for which $p_{\gamma,\mathfrak{D}}$ is well-defined.

Thm (Kontsevich-Soibelman, Gross-Siebert)

Given a scattering diagram $\mathfrak{D}$, there always exists a consistent scattering diagram $\mathfrak{D}'$ s.t. $\mathfrak{D}' \setminus \mathfrak{D}$ consists of only outing walls

Quiver representation

Q : (acyclic) quiver of rank n with: Q_0: set of vertices, Q_1: set of arrows

<u>Def</u> A $\mathbb{C}$-linear representation $V = (V_a, V_\alpha)_{\substack{a \in Q_0 \\ \alpha \in Q_1}}$ of Q is

- each point a in Q_0 is associated a $\mathbb{C}$ vectorspace V_a
- each arrow $\alpha : a \to b$ in Q_1 is associated a $\mathbb{C}$-linear map

$$V_\alpha : V_a \to V_b$$

It is called finite dimensional if each vector space V_a is f.d.

Define dimension vector of V as $\dim V = (\dim V_a)_{a \in Q_0}$

<u>Def</u> A representation V is called <u>indecomposable</u> if $V \neq 0$ and if $V = A \oplus B$ then $A = 0$ or $B = 0$

E.g. $\underset{1}{\bullet} \to \underset{2}{\bullet}$ irreducible rep. are $\mathbb{C} \to 0$, $0 \to \mathbb{C}$, $\mathbb{C} \to \mathbb{C}$

Let $N = \mathbb{Z}Q_0$ and $M = \mathrm{Hom}(N, \mathbb{Z})$, i.e. $n = \#$ vertices in Q.

Now, we define $\{\cdot, \cdot\}$ on N as

$$\{e_i, e_j\} = \{\#\text{arrow from } i \to j\} - \{\#\text{arrow from } j \text{ to } i\}$$

Define a bilinear form $\chi(\cdot,\cdot)$, the Euler form, on N as

$$\chi(d, e) = \sum_{i = Q_0} d_i e_i - \sum_{\alpha = i \to j} d_i e_j \qquad d, e \in N$$

Define $\varepsilon : N \to M$ by $\qquad \varepsilon(d) = \chi(\cdot, d)$

A wall (d, f_d), $d \subseteq n^{\perp}$

i.e. $\langle n, w \rangle = 0 \quad \forall w \in d$

"think w as a stability condition"

A rep E is said to be <u>w-semistable</u> if

- $w(E) = 0$
- every subobj. $B \subseteq E$ satisfies $w(B) \leq 0$

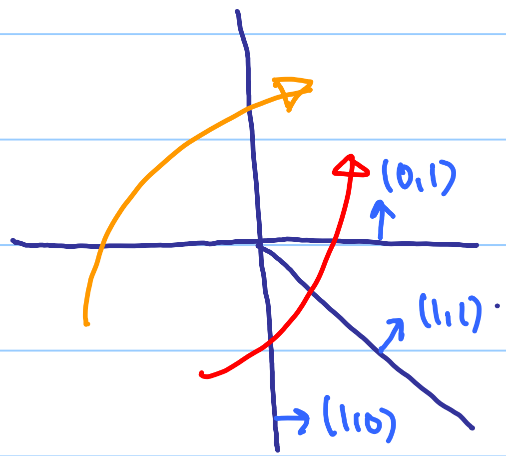
<u>Thm</u> (Bridgeland) Hall alg. scattering diag.

- walls consists of maps $w \in M_{\mathbb{R}}$ s.t.

  $\exists$ w-semistable obj. in $\mathrm{rep}(Q)$

$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$

E.g.



$\rightsquigarrow$

$0 \to \mathbb{C}$

$\mathbb{C} \to \mathbb{C}$

$\mathbb{C} \to 0$

Auslander-Reiten theory

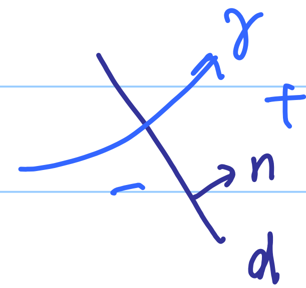
A way to "line up" irreducible rep. of a quiver.

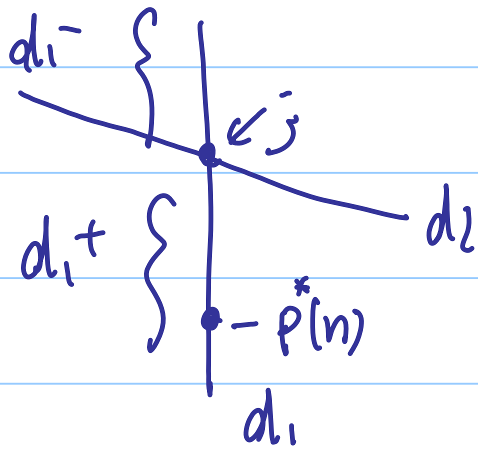
E.g. $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \rightsquigarrow$

$$
\begin{tikzcd}
 & \mathbb{C} \to \mathbb{C} \arrow[dr] & \\
0 \to \mathbb{C} \arrow[ur] & & 0 \to \mathbb{C}
\end{tikzcd}
$$

γ : path in $\mathcal{D}$

Good crossing





$$d_1 = d_1^+ \cup j \cup d_1^-$$

Let D_1, D_2 be the indecomposable rep. associated to d_1, d_2

<u>Thm</u> (c) If γ goes from d_1^+ to d_2, and the two crossing are good, then D_2 is a predecessor of D_1

"reverse the order in Auslander-Reiten quiver"

The original setup from Gross-Hacking-Keel-Kontsevich is 2014 is more general. They have associated scattering diagram to "cluster algebra".

Cluster algebra is defined by Fomin-Zelevinsky in 2000 to understand total positivity in algebraic group and canonical bases in quantum group.

It is a subring of a field of rational functions. The (cluster) variables are generated by an iterative process called mutation.

Fomin-Zelevinsky proved those "iterative" variables can be expressed as Laurent polynomial of initial variables.

It is conjectured that the coeff. in these Laurent poly are non-negative.

GHKK proved it by linking up "theta function" to cluster variables.

can be defined on scattering diag.

Motivation

Broken line: understand Landau-Ginzburg mirror symmetry

Siebert-Carl-Pumperla: made use of broken line to construct Landau-Ginzburg mirrors to varieties with effective anti-canonical bundle.

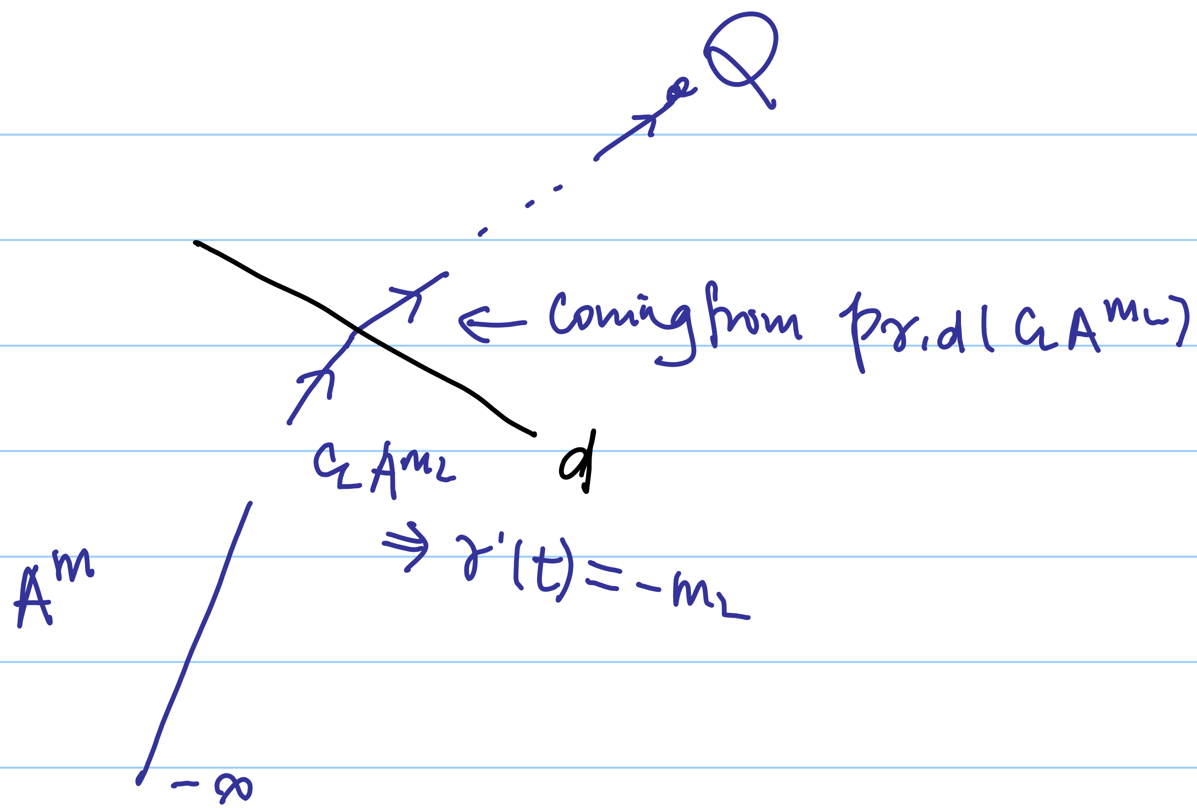
Gross-Hacking-Keel-Siebert.

Theta function is classical theta function for the case of abelian varieties.

Theta function

$\mathfrak{D}$: scattering diag.
$m \in M \setminus \{0\}$
$Q \in \{ m \in M_{\mathbb{R}} \mid \langle m, e_i \rangle > 0 \ \forall i \}$

A broken line with initial slope m and endpt Q is a piecewise linear continuous proper path $\gamma : (-\infty, 0) \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$ with a finite number of domains of linearity.
A monomial $c_L A^{m_L}$ is attached to each domain of linearity $L \subseteq (-\infty, 0)$
s.t.

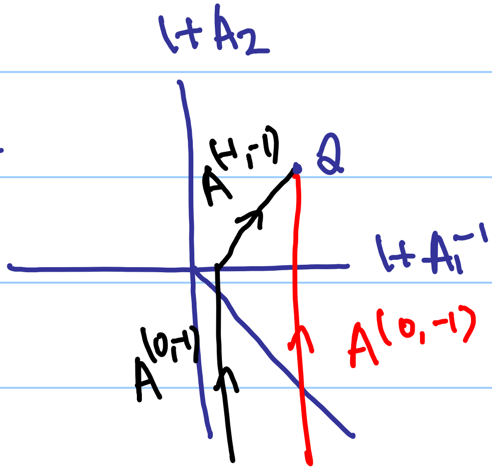


Fix m & Q. Consider all broken line γ with initial slope m & endpt. Q.

Take $c_L A^{m_L}$ of the last domain linearity

$$\vartheta_{m,Q} = \sum_\gamma c_L A^{m_L}$$

E.g.



$m = (0,-1)$

Two broken lines: γ_1, γ_2

$\vartheta_{m,Q} = A^{(0,-1)} + A^{(-1,-1)}$

Thm (GHKK) If $\vartheta_{m,Q}$ is a finite sum, then $\vartheta_{m,Q}$ is an elt. of cluster alg.

Many other proposal for bases of cluser alg.

<u>Thm</u> (Caldero-Chapoton) Let Q be a finite quiver with vertices $1, \dots, n$. and D a f.d. rep$\underline{n}$ of Q with dimension vector d. For $e \in \mathbb{N}$,

denote $Gr(e, D) := \{ E \in \mathrm{mod}(Q) \mid E \subseteq D, \dim(E) = e \}$

Define

$$CC(D) = \frac{1}{A_1^{d_1} \cdots A_n^{d_n}} \sum_{0 \leq e \leq d} \chi(Gr(e, D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)}$$

Then $CC(D) = X_D$ cluster variable obtained from D.

$$CC(D) = \frac{1}{A^d} \sum_{0 \le \ell \le d} \chi(Gr(\ell, D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} \ell_j + \sum_{i \to j} (d_j - \ell_j)}$$

$$= \frac{\prod A_i^{\sum_{i \to j} d_j}}{\prod A_i^{d_i}} \sum_{0 \le \ell \le d} \chi(Gr(\ell, D)) \prod_{i=1}^{n} A_i^{\sum_{j \to i} \ell_j + \sum_{i \to j} -\ell_j}$$

$$= A^{-\varepsilon(d)} \sum_{0 \le \ell \le d} \chi(Gr(\ell, D)) \prod_{i=1}^{n} A_i^{p^*(\ell)}$$

Note: initial slope

$-\varepsilon(d) + p^*(\ell)$ final slope

Thm (Bridgeland) (Hall algebra scattering diag.) Denote $H(Q)$ as Hall alg.

- wall crossing automorphism at a general pt. $w \in d$ is a conjugation by

$$1_{ss}(w) = [\mathcal{M}(w) \to \mathcal{M}]$$

obj. are w-semisimple rep.

Let d be a wall in the cluster complex.

$$1_{ss}(w) = 1 + \sum_{k \geq 1} [\underbrace{BGL_k(D) \to \mathcal{M}}_{\text{think as } D^k}]$$

where D is irr. w-semisimple rep.

Hall algebra broken line.

$\mathfrak{D}$: Hall alg. scattering diag. $m \in M \setminus \{0\}$ and $Q \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D})$

A Hall alg. broken line for m with endpoint Q is a piecewise linear continuous proper path $\gamma : (-\infty, 0] \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$ with a finite number of domains of linearity and there is an elt. in $H(Q) \otimes_{\mathbb{C}} \mathbb{C}[M]$ attached to each domain of linearity $L \subseteq (-\infty, 0)$ of γ.

The path γ and the $[N \to M]A^m$ need to satisfy

- $\gamma(0) = Q$
- if L is the first domain of γ, then $[N \to M]A^m = A^m$.

  This is saying we have $[\cdot \mapsto 0]$ zero representation

- in each domain of linearity $L \subseteq (-\infty, 0]$ the attaching Hall alg. monomial $[N \to M] A^m$

  where $\chi([N \to M]) A^m = \chi([N]) A^{p^*(\dim N) + m}$

  where $\gamma'(t) = -(p^*(\dim N) + m)$ for $t \in L$.

- γ bends only when crossing a wall

  If γ bends from the domain of linearity L to L' when crossing d,

  then $[N \to M] A^m$ is a term in $\Phi_{\mathfrak{D}}(d) \cdot ([N \to M] A^m)$

Thm (c)

Hall alg. theta function

$$\vartheta_m = \mathcal{G}_{F(m)}(D)$$

where objects in $\mathcal{G}_{F(m)}(D)$ are repns E s.t.
for any subobj. $F \subseteq E$, $m(F) \leq 0$.
Furthermore E is equipped with an inclusion into D.

apply Euler char χ

$$\longrightarrow \vartheta_m = A^{-\varepsilon(D)} \sum_{0 \leq e \leq d} \chi(Gr(e, D))\, A^{p^*(e)}$$

reprove the CC-formula

Setting: γ: broken line with endpt in positive chamber

good crossing over outgoing walls

with initial slope $\varepsilon(D)$ $\quad$ D indecomp.

final slope $\varepsilon(D) - p^*(E)$ $\quad$ $E \subseteq D$ subrep

Assume γ bends over the

walls $\subseteq f_1^\perp, \ldots, f_s^\perp$ with multi. $\lambda_1 \ldots \lambda_s$

Corresp. to indecomp $F_1 \ldots F_s$

First bending

Thm $pr_{f_1^\perp}(A^{-\varepsilon(d)}) = \mathcal{G}_1 A^{-\varepsilon(d)}$

where obj. in $\mathcal{G}_1$ are $[F_1^\lambda \to D]$

with no kernel of dim vector f_1

Apply χ $\rightsquigarrow$ get $\sum_\lambda Gr(\lambda, Hom(F_1 D))$

which agrees with usual wall-crossing

Since mult. λ_1, define $V_1 := F_1^{\lambda_1}$

Second bending

Thm (C) $pr_{\lambda, f_2^{\perp}}\left(\mathcal{G}_1 A^{-\varepsilon(D)}\right) = \mathcal{G}_2 A^{-\varepsilon(d)}$

where Poincare poly. of $\mathcal{G}_2$ is the same as that of

$Gr(\lambda, \mathrm{Hom}(F_2, D/F_1) - \mathrm{Ext}^1(F_1, F_2)) \times A^{\lambda \mathrm{Ext}^1(F_1, F_2)}$

also get a filtration

$0 \subseteq V_1 \subseteq V_2$

where $\dim V_2 = \dim F_1^{\lambda_1} + \dim F_2^{\lambda_2}$

k-th bending

from $(k-1)$, have $\quad 0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1}$, $\quad V_i/V_{i-1} = F_i^{\lambda_i}$

get $\mathcal{G}_k$ where Poincare poly are of the form

$$Gr\left(\lambda_k \, Hom(F_k, D/V_{k-1}) - Ext(V_{k-1}, F_k)\right) \times \mathbb{A}^{\lambda_k Ext(V_{k-1}, F_k)}$$

& $\quad 0 \subseteq V_1 \subseteq \cdots \subseteq V_{k-1} \subseteq V_k$

$\dim V_k = V_{k-1} + \lambda_k \dim F_k$

till the last wall f_s^+ get $D = V_s$

have Harder-Narasimham property in dim 2.

Joint work with Travis Mandel (work in progress)

Fix $Q \in M_{\mathbb{R}}$.

Now given $\vartheta_{p_1}, \dots, \vartheta_{p_s}$, what is $\vartheta_{p_1} \times \cdots \times \vartheta_{p_s}$?

E.g. $\vartheta_{p_1} \cdot \vartheta_{p_2} = \sum_q \alpha(p_1, p_2, q)\, \vartheta_q$

Final monomial

[GHKK 14] $\alpha(p_1, p_2, q) = \sum_{\substack{(\gamma_1, \gamma_2) \\ I(\gamma_i) = p_i,\ b(\gamma_i) = z \\ F(\gamma_1) + F(\gamma_2) = q}} c(\gamma_1)\, c(\gamma_2)$

i.e. broken line with initial slope & final slope satisfying these condition

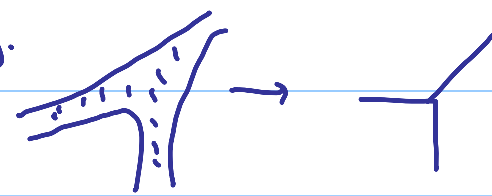
[Mandel] Interpret the coefficient of z^n in $\vartheta_{p_1} \cdots \vartheta_{p_n}$ as Block-Göttsche weighted count of marked $\underbrace{\text{tropical curve}}$.

" $\nearrow$ "

weighted finite tree with balancing condition

limit of classical obj.

e.g.



Now: Interpret of the coeff. using the machinery above.

THANK YOU!